Attractors in real-time dynamical systems

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Abstract

We introduce cluster sets, periodic orbits and attractors for real-time dynamical systems. We furthermore show some basic results considering attractors in topologically transitive systems.

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Author notice

This is an article about cluster points and attractors in real-time dynamical systems that resulted rather spontaneously from personal reflections on the subject in 2011. It is not peer-reviewed! The article assumes basic knowledge of topology, as provided for example in [2]. Most of the definitions were taken from [3], albeit with some modifications. A more thorough elaboration on the subjects mentioned can be found in [5].

1 Attractors in real-time dynamical systems

1.1 Preliminaries

1.1.1 Definition: Dynamical system

An abelian semi-group $(G, +)$, to be called **time domain**, acting on a non-empty set $X \neq \emptyset$ is called a **semi-flow** on X. We call (X, G) a **dynamical system**. If G is a group, we call G a **flow** and (X, G) an **invertible** dynamical system. For $x \in X$ and $g \in G$, we interpret $g(x)$ as point *reached after time* g, starting from the *start point* x.

If X is a topological space and each g acts on X as an continuous (open, closed) mapping, we call the system **space-continuous** (**space-open**, **space-closed**). Note that for invertible systems space-continuity is equivalent to space-openness and equivalent to space-closeness.

If X is a topological space, G a topological group and the action $G \times X \to X$, $(g, x) \mapsto g(x)$ is continuous with respect to the product topology on $G \times X$, then (G, X) shall be called a **continuous** dynamical system. It shall be called **time-continuous** if the action $(g, x) \mapsto g(x)$ is continuous in time for every fixed start point x. Note that continuous systems are space- and time-continuous, with the reverse not always being true.

For any $x \in X$ we call the set $Gx := (g(x))_{g \in G}$ the **orbit** of x under the semi-flow G. A point $x \in X$ is called **periodic**, if $g(x) = x$ for some time $g \neq \mathrm{Id}$. An orbit Gx is called **periodic** if it contains a periodic point.

A point $x_0 \in X$ is called a **fixed point** of the system if $g(x_0) = x_0$ for all $g \in G$. We call a set $A \subseteq X$ **invariant** to the semi-flow, if $g(A) = A$ for all $g \in G$. Note that for invertible systems this is equivalent to $g(A) \subseteq A$ for all $g \in G$.

1.1.2 Definition: Real-time system

We call a dynamical system (X, G) a **real-time** system if $G = [0, \infty)$ or $G = \mathbb{R}$. In that case we write $G_t : X \to X$ for the mapping induced on X by $t \in \mathbb{R}$. For any point $x \in X$, we call $(G_t(x))_{t \geq 0}$ the **future orbit** of x under the semi-flow. It is called **completely periodic**, if $G_t(x) = x$ for some $t > 0$.

We call a set $A \subseteq X$ **forward invariant** to the semi-flow if $G_t(A) \subseteq A$ for all $t \geq 0$. Note that for invertible real-time systems, invariance of a set in $(X, (G_t)_{t \geq 0})$ is equivalent to the invariance of the set in $(X, (G_t)_{t \in \mathbb{R}})$.

1.1.3 Definition: Topologically transitive systems

A dynamical system (X, G) on the topological space X is called **topologically transitive** if for every pair of non-empty open sets U and V in X, there exists some $g \in G$ such that $g(U) \cap V \neq \emptyset$. We call an invariant set $A \subseteq X$ **topologically transitive** if the restriction $(A, G\big|_A)$ constitutes a topologically transitive dynamical system.

Remarks:

(i) If (X, G) is a real-time, invertible dynamical system, then $(X, (G_t)_{t \geq 0})$ is topologically transitive if and only if $(X, (G_t)_{t \in \mathbb{R}})$ is topologically transitive.

(ii) Some authors choose to define topological transitivity as the condition of the existence of a dense orbit in X. In general topological spaces, neither definition implies the other.

1.1.4 Definition: Convergence towards sets

Let X be a topological space and $A \subseteq X$ some set. We say that a net[1] (Moore-Smith sequence) $(x_i)_{i \in I}$ **converges** towards A if for every neighborhood B of A the sequence is eventually in B.

[1] Sequences indexed over directed sets. We summarize:

- A function $f : X \to Y$ between topological spaces X, Y is continuous if an only if for every net $(x_\alpha)_\alpha \subseteq X$ converging

Remarks:

(i) Let (X, d) be a metric space and $A \subseteq X$. If the net $(x_i)_i$ converges towards A, then $d(A, x_i) \xrightarrow{i} 0$.

(ii) The inverse statement is true provided that A is a compact set.

1.1.5 Definition: Attracting sets

Let (X, G) be a real-time dynamical system on the topological space X and $A \subseteq X$ some set. A neighborhood $B \subseteq X$ of A is called a **neighborhood of attraction** if for all $y \in B$ one has $G_t(y) \overset{t\to\infty}{\longrightarrow} A$. The union of all orbits eventually converging to A is called its **basin of attraction**.
A set $A \subseteq X$ is called **Lyapunov stable**, if for each neighborhood $B \subseteq X$ of A there exists a neighborhood $C \subseteq X$ of A such that $G_t(C) \subseteq B$ for all $t \geq 0$. It is called **locally attracting** if it has a neighborhood of attraction and **globally attracting** if the whole space X is a neighborhood of attraction for A.

Remark: The above notions of stability typically only make sense for forward invariant sets A.

1.1.6 Definition: Forward fixed points

Let (X, G) be a real-time dynamical system. A point $x_0 \in X$ is called a **forward fixed point** if $G_t(x) = x$ for all $t \geq 0$. Now let X be a topological space. A forward fixed point x_0 is called **Lyapunov stable** if $\{x_0\}$ is Lyapunov stable, that is if for any neighborhood U of x_0 there exists a neighborhood V of x_0, such that $G_t(V) \subseteq U \ \forall t \geq 0$. It is called **locally attracting** if there exists a neighborhood U of x_0 such that for all $x \in U$ one has $G_t(x) \overset{t\to\infty}{\longrightarrow} x_0$. It is called **globally attracting** if for all $x \in X$ one has $G_t(x) \overset{t\to\infty}{\longrightarrow} x_0$.
It is called **asymptotically stable** if it is both Lyapunov stable and locally attracting, otherwise it is called **unstable**. It is called **globally asymptotically stable** if it is Lyapunov stable and globally attracting.

1.1.7 Definition: Future cluster and limit set

Let (X, G) be a real-time dynamical system on the topological space X and $x \in X$. A point $y \in X$ is called a **future cluster point** of x if for every neighborhood U of y the future orbit $(G_t(x))_{t\geq 0}$ is frequently[2] in U. It is called a **future limit point** (**ω-limit point**) of x if $G_{t_n}(x) \overset{n\to\infty}{\longrightarrow} y$ for some sequence $0 \leq t_1 < t_2 < .. \to \infty$. We shall write $G_{\text{cl}}(x)$ and $G_{\text{lim}}(x)$ for the set of all future cluster points and all future limit points of x respectively, to be called **future cluster set** and **future limit set** of x.

1.1.8 Lemma on closures and interiors of invariant sets

Let (X, G) be a real-time dynamical system on the topological space X and $A \subseteq X$ some set. Then:

1. If the system is space-continuous and A forward invariant, the closure $\overline{A}$ is forward invariant.
2. If the system is space-open and A forward invariant, the interior A^o is forward invariant.
3. If the system is space-continuous and space-closed and A invariant, the closure $\overline{A}$ is invariant.
4. If the system is invertible and space-continuous and A (forward) invariant, then both A^o and $\overline{A}$ are (forward) invariant.

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towards $x \in X$, the net $(f(x_\alpha))_\alpha$ converges towards $f(x)$.

- A space X is compact if and only if every net $(x_\alpha)_\alpha \subseteq X$ has a subnet with a limit in X.
- A subset A of a space X is closed if and only if, every limit of a net with elements in A, is again in A.
- A net has a limit if and only if all of its subnets have limits. In that case, every limit of the net is also a limit of every subnet.
- If X is Hausdorff, then every net in X has at most one limit.
- A net in a product space has a limit if and only if each projection of the net has a limit, equal to the projection of the limit.
- Every sequence is a net. But subnets of sequences are not always subsequences!
- Subnets of subnets of a net are also subnets of that net.
- If a net is frequently in some set A, then it has a subnet which is eventually in A.

See [2] for more on nets and their connection to the topology of a space.

[2] A net $(x_\alpha)_{\alpha\in A}$ is **frequently** in some set U if for every $a \in A$ there exists a $b \in A$ with $x_b \in U$.

5. If the system is invertible and space-continuous and A invariant, then the border ∂A is invariant.

Proof:

1. It suffices to show that $\overline{A} \subseteq G_t^{-1}(A)$ for $t \geq 0$, since then $G_t(\overline{A}) \subseteq G_t(G_t^{-1}(\overline{A})) \subseteq \overline{A}$. Indeed, $G_t^{-1}(\overline{A})$ is by continuity of G_t a closed set including A.
2. Since $G_t(A^o)$ is for each $t \geq 0$ an open set included in A, one has indeed $G_t(A^o) \subseteq A^o$.
3. In a similar way as in (1), one shows that $G_t(A) \subseteq A$ for all $t \geq 0$. Furthermore, $G_t(\overline{A})$ is for all $t \geq 0$ a closed set containing $G_t(A)$ and thus A, hence also $\overline{A}$. If the system is invertible, the invariance of A in $(X, (G_t)_{t\geq 0})$ implies the invariance of the set in $(X, (G_t)_{t\in\mathbb{R}})$.
4. Note that the system is because of invertibility also space-open and space-closed. By the previous points, it suffices to show that $G_t(A^o)$ is invariant, provided that A is invariant. Since $G_t(A^o)$ is for each $t \in \mathbb{R}$ an open set included in A, one has $G_t(A^o) \subseteq A^o$ for all $t \in \mathbb{R}$. As the system is invertible, this already implies the invariance of A.
5. For each $t \in \mathbb{R}$, $G_t : X \to X$ is bijective and by the previous points one has $G_t(A^o) = A^o$ and $G_t(\overline{A}) = \overline{A}$. Thus $\partial A = \overline{A} \setminus A^o = G_t(\overline{A}) \setminus G_t(A^o) = G(\overline{A} \setminus A^o) = G_t(\partial A)$.

□

1.1.9 Lemma: Characterization of locally attracting sets

Let (X, G) be a real-time, space-continuous dynamical system on the topological space X and $A \subseteq X$ some set. Then A is locally attracting if and only if its basin of attraction B is open. In that case one has $B = \bigcup_{t\geq 0} G_t^{-1}(U)$ for any arbitrary neighborhood of attraction U.

Proof: Direction "$\Leftarrow$" is trivial. Now suppose A to be locally attracting and let U be some neighborhood of attraction for A. Set $\Omega := \bigcup_{t\geq 0} G_t^{-1}(U)$ and let B be then basin of attraction of A. Then $B \subseteq \Omega$, since every orbit starting in B eventually plunges into U. But by construction of Ω, every orbit starting in Ω passes by U and, since U is a neighborhood of attraction, eventually converges to A. Thus also $\Omega \subseteq B$. Since U can be chosen to be open, $\Omega = B$ is open by continuity of each G_t.

□

1.1.10 Lemma on stable sets

Let (X, G) be a real-time, space-open dynamical system on the topological space X and $A \subseteq X$ a Lyapunov stable, locally attracting set. Then for each neighborhood B of A there exists a forward invariant neighborhood of attraction for A, included in B.

Proof: Let U be some neighborhood of attraction for A. We can suppose that $U \subseteq B$, otherwise $U \cap B$ would do the job. Note $\mathcal{V}$ the set of all neighborhoods V of A such that $G_t(V) \subseteq U$ for all $t \geq 0$. By the Lyapunov stability of A, $\mathcal{V}$ is not empty. Note that if $V \in \mathcal{V}$, then also $G_t(V) \in \mathcal{V}$ for all $t \geq 0$ by openness of G_t. Thus, the set $\Omega := \bigcup_{V\in\mathcal{V}} V$ is a forward invariant neighborhood of A. Moreover, since $\Omega \subseteq U$, it is a neighborhood of attraction.

□

1.2 Future cluster sets and periodic orbits

1.2.1 Lemma: Characterization of fixed and periodic points

Let (X, G) be a real-time dynamical system on the Hausdorff space X and $x \in X$. Then:

1. x is a forward fixed point if and only if there exists an $\varepsilon > 0$ such that $G_t(x) = x$ for all $0 \leq t \leq \varepsilon$.
2. Suppose the system is time-continuous. Then x is a periodic, non forward fixed point if and only if there exists a smallest $t_0 > 0$ such that $G_{t_0}(x) = x$. That time t_0 is called the **period** of x.

Proof:

1. Follows from the definition and the fact that G acts as a semi-group.

2. Direction "$\Leftarrow$" follows from the relevant definitions. Let x be non forward fixed and periodic. We shall show that the set

$$T_x := \{t > 0 : G_t(x) = x\} \neq \emptyset \tag{1.1}$$

is closed in $\mathbb{R}$, so that $t_0 := \min T_x$ satisfies the mentioned properties[3].

Let $0 < t_0 \notin T_x$, that is $G_{t_0}(x) \neq x$. Then as X is Hausdorff, by time-continuity of the system there exists a neighborhood $U \subseteq (0, \infty)$ of t_0 such that $G_t(x) \neq x$ for all $t \in U$. This shows that $U \cap T_x = \emptyset$ and that T_x is closed in $(0, \infty)$. Left to show is that, 0 is not a limit point of T_x. Indeed, suppose that there exists $(t_n)_n \subseteq T_x$ such that $t_n \overset{n\to\infty}{\longrightarrow} 0$. Then each time $t > 0$ can be approximated by integer multiples of adequately small members of $(t_n)_n$, that is t is a limit point of T_x and thus within T_x. This would mean that x is a forward fixed point, a contradiction.

□

1.2.2 Theorem: Properties of future cluster and limit sets

Let (X, G) be a real-time dynamical system on the topological space X and $x \in X$. Then:

1. $G_{\lim}(x) \subseteq G_{\mathrm{cl}}(x)$. The reverse is true provided that X is first-countable.
2. The future cluster set $G_{\mathrm{cl}}(x)$ is closed in X.
3. If the system is space-continuous, then $G_{\mathrm{cl}}(x)$ as well as $G_{\lim}(x)$ are forward invariant sets.
4. If the system is space-continuous and invertible, then $G_{\mathrm{cl}}(x)$ as well as $G_{\lim}(x)$ are invariant sets.
5. Let X be Hausdorff and $K \subseteq X$ be compact. If the future orbit $(G_t(x))_{t\geq 0}$ converges to K, then $G_{\mathrm{cl}}(x)$ is a compact subset of K.
6. Let the system be time-continuous, X be Hausdorff and $K \subseteq X$ be compact, enclosed by a compact neighborhood. If the future orbit $(G_t(x))_{t\geq 0}$ converges to K, then $G_{\mathrm{cl}}(x)$ is a non-empty, connected, compact subset of K.
7. Let the system be time-continuous, X be Hausdorff and $K \subseteq X$ be compact, enclosed by a compact neighborhood. If the future cluster set $G_{\mathrm{cl}}(x)$ is non-empty and completely within K, then the future orbit of x converges to K.

Proof:

1. Trivial.
2. Let $y \notin G_{\mathrm{cl}}(x)$, then there exists an open neighborhood U of y such that $(G_t(x))_{t\geq 0}$ is eventually in U^c. As U is a neighborhood for all of its points, $x \in U \subseteq X \setminus G_{\mathrm{cl}}(x)$, proving that $G_{\mathrm{cl}}(x)$ is closed.
3. Let $y \in G_{\mathrm{cl}}(x)$ and $\tau \geq 0$, we show that $G_\tau(y) \in G_{\mathrm{cl}}(x)$. For any neighborhood U of $G_\tau(y)$ we can by continuity of G_τ choose a neighborhood V of y such that $G_\tau(V) \subseteq U$. Then for any $t \geq 0$ there exists an $s \geq t$ such that $G_s(x) \in V$, hence $G_{s+\tau}(x) \in U$, which was to be shown.

   Now let $y \in G_{\lim}(x)$, that is $y = \lim_{n\to\infty} G_{t_n}(x)$ for some $0 \leq t_1 < t_2 < .. \to \infty$. Then for any $\tau \geq 0$ one has by continuity $G_\tau(y) = \lim_{n\to\infty} G_\tau(G_{t_n}(x)) = \lim_{n\to\infty} G_{t_n+\tau}(x)$, which shows that $G_\tau(y) \in G_{\lim}(x)$.
4. Similar to (3), using the characterization of invariance mentioned in 1.1.1.
5. Let $y \in G_{\mathrm{cl}}(x)$ be some future cluster point of x. Then every neighborhood U of y intersects every neighborhood V of K. Since both $\{y\}$ and K are compacts, by lemma A.0.4 $y \in K$, that is $G_{\mathrm{cl}}(x) \subseteq K$. Since by (2) $G_{\mathrm{cl}}(x)$ is closed, it is compact.

[3] It is easy to see that the period t_0 is a generator of the additive semi-group $T_x \cup \{0\}$.

6. Suppose $\widetilde{K} \subseteq X$ to be some compact neighborhood of K and $G_{\mathrm{cl}}(x)$ not to be connected. Then $G_{\mathrm{cl}}(x)$ would consist of two disjoint, non-empty compact parts $K_1, K_2 \subseteq K$. By lemma A.0.4 there would exist two disjoint, open (in X) sets U_1, U_2, each enclosing one of the two parts. We may of course assume that $U_1, U_2 \subseteq \widetilde{K}$. As $(G_t(x))_{t\geq 0}$ converges to K, it would eventually have to be within $\widetilde{K}$. As x has cluster points in K_1 as well as K_2, it passes from U_1 to U_2 and vice versa an infinite number of times. As U_1, U_2 are disjoint open sets and the orbit $G(x) : [0,\infty) \to \mathbb{R}$ continuous, it would have to exit the set $U_1 \cup U_2$ and pass by $R := \widetilde{K} \setminus (U_1 \cup U_2)$ an infinite number of times. As R is a compact set, the orbit would have at least one cluster point in R, a contradiction to $G_{\mathrm{cl}}(x)$ being enclosed by $U_1 \cup U_2$.

   Clearly $G_{\mathrm{cl}}(x)$ is non-empty since the orbit is eventually within the compact $\widetilde{K}$. The rest is given by point (5).

7. Let $\widetilde{K} \subseteq X$ be some compact neighborhood of K and U some arbitrary neighborhood of K. We show that $\Gamma_x := (G_t(x))_{t\geq 0}$ is eventually within U. We can w.l.o.g. assume $U \subseteq \widetilde{K}$ and U to be open. Now suppose Γ_x to be frequently in U^c. As Γ_x has a cluster point in K, it is also frequently in U. By time-continuity, it thus passes frequently by ∂U. As $\partial U \subseteq K$ is compact, Γ_x possesses in ∂U a cluster point. As $\partial U \cap U = \emptyset$, that cluster point is not within K, a contradiction!

□

1.2.3 Corollary for compact systems

Let (X, G) be a real-time, time- and space-continuous dynamical system on the compact Hausdorff space X. Then for every $x \in X$, the future cluster set $G_{\mathrm{cl}}(x)$ is a non-empty, connected, compact, forward invariant set.

Proof: Trivially, the orbit $(G_t(x))_{t\geq 0}$ converges to the compact X. By theorem 1.2.2 follows the affirmation. □

Note: The future cluster set G_{cl} need not be an orbit itself[4]. However, if $G_{\mathrm{cl}}(x)$ is a completely periodic future orbit of the system and disjoint from the orbit $(G_t(x))_{t\geq 0}$, it is called a **future limit cycle** (**ω-limit cycle**) (of x).

1.2.4 Theorem: Characterization of periodic orbits

Let (X, G) be a real-time, time-continuous, space-continuous dynamical system on the Hausdorff space X. Then a future orbit $\Gamma := (G_t(x))_{t\geq 0}$ is compact if and only if it is periodic. In that case, there exists a *smallest time* $t_0 \geq 0$ such that $G_{t_0}(x)$ is periodic. Furthermore, $G_{\mathrm{cl}}(x)$ is the future orbit of $G_{t_0}(x)$ and consists of all periodic points of Γ. If furthermore the system is invertible, then Γ is completely periodic.

Proof: The proof is inspired by [5]. See [6] for a generalization. Direction "$\Leftarrow$" is trivial. Suppose now $\Gamma := (G_t(x))_{t\geq 0}$ to be compact, so that one has $\emptyset \neq G_{\mathrm{cl}}(x) \subseteq \Gamma$. Suppose Γ to be non-periodic. The mapping $\mathbb{R}_+ := [0,\infty) \to X$, $t \mapsto G_t(x)$ is then injective. By time-continuity of the system, every *segment* $\Gamma_n := (G_t(x))_{t\leq n}$ is a compact and thus closed subset of Γ. As Γ contains $G_{\mathrm{cl}}(x) \neq \emptyset$, there exists a point $y \in \Gamma$ such that each neighborhood of y is visited by the future orbit at arbitrarily late times, that is intersects $(G_t(x))_{t>n} = \Gamma \setminus \Gamma_n$ for all $n \in \mathbb{N}$. Thus $y \in \overline{\Gamma \setminus \Gamma_n}$ for all $n \in \mathbb{N}$. As $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$, by lemma A.0.5 Γ can not be locally compact Hausdorff, a contradiction! Thus Γ is indeed periodic.
Now let

$$T_{\mathrm{cl}} := \{t \geq 0 : G_t(x) \in G_{\mathrm{cl}}(x)\}\,. \tag{1.2}$$

We have already seen that $T_{\mathrm{cl}} \neq \emptyset$. We show that T_{cl} is closed in $\mathbb{R}_+ := [0,\infty)$, by showing that its complement $T_{\mathrm{cl}}^c := \mathbb{R}_+ \setminus T_{\mathrm{cl}}$ is open in $\mathbb{R}_+$. Let $0 \leq t_0 \notin T_{\mathrm{cl}}$, that is $G_{t_0}(x) \notin G_{\mathrm{cl}}(x)$. By lemma 1.2.2(2), $G_{\mathrm{cl}}(x)$ is closed in X, so that there exist an open neighborhood $U \subseteq X$ of $G_{t_0}(x)$ not intersecting $G_{\mathrm{cl}}(x)$. As the system is time-continuous, there exists an open neighborhood $\widetilde{U} \subseteq \mathbb{R}_+$ of t_0 such that $G_t(x) \in U$ for all $t \in \widetilde{U}$, that is $\widetilde{U} \subseteq T_{\mathrm{cl}}^c$.
Thus the minimum $t_0 := \min T_{\mathrm{cl}}$ exists. We note $\Gamma_0 := (G_t(x))_{t\geq t_0}$ the future orbit of $G_{t_0}(x)$. By lemma 1.2.2(3) and space-continuity of the system, future cluster sets are forward invariant, so that $\Gamma_0 \subseteq G_{\mathrm{cl}}(x)$. By choice of t_0, one has actually $G_{\mathrm{cl}}(x) = \Gamma_0$. Indeed, if $G_{t_1}(x) \in G_{\mathrm{cl}}(x) \setminus \Gamma_0$ for some $t_1 \geq 0$, this would imply $t_1 < t_0$, a contradiction to the minimality of t_0.

[4] It could for example consist of the whole space, if the future orbit $(G_t(x))_{t\geq 0}$ was dense in X.

We now show that the periodic points of Γ are exactly its cluster points. Indeed, each periodic point is obviously a cluster point. Inversely, let $x_1 := G_{t_1}(x)$ be a cluster point of Γ for some time $t_1 \geq 0$. As Γ is periodic, the future orbit $\Gamma_1 := (G_t(x))_{t \geq t_1+1}$ is actually given by $(G_t(x))_{t_1+1 \leq t \leq T}$ for some $T \geq 0$ adequately large. By time-continuity of the system, Γ_1 is therefore compact. As $G_{\text{cl}}(x) = G_{\text{cl}}(G_{t_1+1}(x)) \subseteq \Gamma_1$, x_1 is in Γ_1, that is, re-attained at some later time $t > t_1$ and thus periodic.
Now suppose the system to be invertible and $t_0 \geq 0$ such that $x_0 := G_{t_0}(x)$ is a periodic point, with $G_T(x_0) = x_0$ for some $T > 0$. Then $x = G_{-t_0}(x_0) = G_{-t_0}(G_{t_0+T}(x)) = G_T(x)$, showing that x is periodic itself.

□

1.2.5 Theorem: Characterization of limit cycles

Let (X, G) be a real-time, time-continuous, space-continuous dynamical system on the locally compact Hausdorff space X. Let $\Gamma_x := (G_t(x))_{t \geq 0}$ be a completely periodic future orbit. Let $\Gamma_y := (G_t(y))_{t \geq 0}$ be some future orbit and $G_{\text{cl}}(y)$ the future cluster set of y. Then the following are equivalent:

1. Γ_y converges towards the set Γ_x.

2. $G_{\text{cl}}(y) = \Gamma_x$.

3. $\emptyset \neq G_{\text{cl}}(y) \subseteq \Gamma_x$.

If $\Gamma_x \cap \Gamma_y = \emptyset$, then Γ_x is a future limit cycle of y if and only if any of the above holds.

Proof: Note that as the system is time-continuous and Γ_x periodic, Γ_x is compact as a set.

(1)⇒(2): As X is locally compact and Hausdorff, there exists a compact neighborhood of Γ_x. Thus by 1.2.2(6), $G_{\text{cl}}(y)$ is a non-empty, compact subset of Γ_x. We show that actually $\Gamma_x \subseteq G_{\text{cl}}(y)$. Fix some future cluster point $x_0 \in \Gamma_x$ of Γ_y. For any $t_0 \geq 0$ and any neighborhood V of $G_{t_0}(x_0)$, let $U \subseteq X$ be a neighborhood of x_0 such that $G_{t_0}(U) \subseteq V$. Then, as Γ_y is frequently in U, it is also frequently in V, so that $G_{t_0}(x_0)$ is also a future cluster point of y. This shows that $\Gamma_x = G_{\text{cl}}(y)$.

(2)⇒(3): Trivial.

(3)⇒(1): As X is locally compact and Hausdorff, Γ_x is enclosed by a compact neighborhood. Thus, lemma 1.2.2(7) applies.

The last affirmation follows from the definition of a future limit cycle.

□

1.3 Attractors

1.3.1 Definition: Attractor

Let (X, G) be a real-time dynamical system on the topological space X. A compact, invariant set $\emptyset \neq A \subseteq X$ is called an **attractor** if there exists a forward invariant neighborhood $U \subseteq X$ of A, such that $A = \bigcap_{t \geq 0} G_t(U)$. The attractor is called **global**, if $A = \bigcap_{t \geq 0} G_t(X)$. It is called **minimal** if it does not strictly contain any other attractors.

Remarks:

(i) A is locally maximal in the following sense: Every invariant set $B \subseteq U$, that is satisfying $G_t(B) = B \ \forall t \geq 0$, is included in A.

(ii) In particular if (X, G) is invertible: Every orbit completely included in U lies in fact completely within A.

(iii) If (X, G) is invertible, the invariance of A follows from the rest of the definition.

(iv) For any other set V between A and U, that is $A \subseteq V \subseteq U$, one has $A = \bigcap_{t \geq 0} G_t(V)$ as well.

(v) If A is an attractor in (X, G) and $B \subseteq X$ some forward invariant set containing A, then A is also an attractor for the smaller system $(B, (G_t\big|_B)_{t \geq 0})$.

(vi) A global attractor is unique and includes all other attractors of the system.

1.3.2 Theorem: Stable sets as attractors

Let (X, G) be a real-time, space-open and space-continuous dynamical system on the locally compact Hausdorff space X. Then every compact, invariant, Lyapunov stable and locally attracting set $A \subseteq X$ is an attractor. As a special case, every asymptotically stable, forward fixed point is an attractor.

Proof: Let V be an open neighborhood of attraction for A. Since A is compact and X locally compact Hausdorff, there exists a compact neighborhood K of A, included in U. By lemma 1.1.10, there exists a forward invariant neighborhood U of attraction for A, included in K. We show that $A = \bigcap_{t\geq 0} G_t(U)$. Obviously $A \subseteq \bigcap_{t\geq 0} G_t(U)$ since A is invariant, so that it suffices to show $\bigcap_{t\geq 0} G_t(U) \subseteq A$. Since the space is Hausdorff, it suffices to show that for any open neighborhood B of A one has $\bigcap_{t\geq 0} G_t(U) \subseteq B$. By lemma 1.1.10, it suffices to consider forward invariant B-s. For such a B and $m \in \mathbb{N}$ let

$$V_{B,m} := \{x \in V : G_m(x) \in B\} = G_m^{-1}(B) \cap V. \tag{1.3}$$

By positive invariance of B one has $V_{B,m} \subseteq V_{B,m+1}$. By continuity of G_m each $V_{B,m}$ is open. Furthermore, by choice of V one has $V \subseteq \bigcup_{m\in\mathbb{N}} V_{B,m}$. Since K is compact and included in V, there exists an M_B such that $K \subseteq \bigcup_{m=1}^{M_B} V_{B,m} = V_{B,M_B}$. Since U is included in K, also $U \subseteq V_{B,M_B}$, that is $G_{M_B}(U) \subseteq B$. Thus $\bigcap_{t\geq 0} G_t(U) \subseteq G_{M_B}(U) \subseteq B$.

□

1.3.3 Lemma on continuous functions

Let T, X, Y be topological spaces and $G : T \times X \to Y,\ (t, x) \mapsto G_t(x)$ continuous. Let $U \subseteq Y$ be open and $I \subseteq T$ be compact. Then the set

$$\Omega := \bigcap_{t\in I} G_t^{-1}(U) \tag{1.4}$$

is open in Y.

Proof: This proof uses the characterization of continuity and compactness via nets. Let $x \in \Omega$ and note $\mathcal{V}$ the system of neighborhoods of x, considered as a directed set with respect to the inclusion. Now suppose that for every neighborhood $V \in \mathcal{V}$ of x, one has $V \nsubseteq \Omega$, that is, there exist $(t_V, x_V) \in I \times V$ such that $G_{t_V}(x_V) \notin U$. Note that the net $(x_V)_{V\in\mathcal{V}}$ converges towards x. As I is compact, there exists a subnet $(t_{\tilde{V}})_{\tilde{V}}$ of $(t_V)_{V\in\mathcal{V}}$ that converges towards some $t \in I$. Since the subnet $(x_{\tilde{V}})_{\tilde{V}}$ still converges towards x, the subnet $(t_{\tilde{V}}, x_{\tilde{V}})_{\tilde{V}}$ converges towards (t, x). By continuity of G, this implies that $(G_{t_{\tilde{V}}}(x_{\tilde{V}}))_{\tilde{V}}$ converges towards $G_t(x) \in U$. But this is a contradiction, since $G_{t_{\tilde{V}}}(x_{\tilde{V}}) \in U^c$ and U^c is closed. Thus, $x \in V \subseteq \Omega$ for some neighborhood V of x.

□

1.3.4 Lemma on trapping neighborhoods

Let (X, G) be a real-time, continuous dynamical system on the Hausdorff space X and $A \subseteq X$ some forward invariant set. Let $K \subseteq X$ be a compact neighborhood of A such that $A \supseteq \bigcap_{t\geq 0} G_t(K)$. Then there exists a forward invariant, open neighborhood Ω of A such that $A \subseteq \Omega \subseteq K^o$.

Proof: The following proof is a generalization of the proof found in [1] for the discrete case to the real-time case. For $t_0 \geq 0$ define

$$\Omega_{t_0} := \bigcap_{0\leq t\leq t_0} G_t^{-1}(K^o). \tag{1.5}$$

as the set of start-points in K^o staying within K_o up to time t_0. By lemma 1.3.3, each Ω_{t_0} is open. As $A \subseteq K^o$ is forward invariant, $A \subseteq \Omega_{t_0}$. Moreover, $G_\tau(\Omega_{t_0}) \subseteq \Omega_{t_0-\tau}$ for every $0 \leq \tau \leq t_0$ and $\Omega_{t_0} \supseteq \Omega_{t_1}$ for every $0 \leq t_0 \leq t_1$. We define

$$\Omega := \bigcap_{t_0\geq 0} \Omega_{t_0} \tag{1.6}$$

as the set of start-points in K^o always staying within K^o. Note that $A \subseteq \Omega \subseteq K^o$. Furthermore, Ω is forward invariant, since $G_\tau(\Omega) \subseteq \bigcap_{t_0\geq\tau} \Omega_{t_0-\tau} = \Omega$ for every $\tau \geq 0$.
Claim: Either $\Omega_{t_0} \supsetneq \Omega_{t_1}$ for all $0 \leq t_0 < t_1$ or $\Omega = \Omega_{t_0}$ for some $t_0 \geq 0$.

Proof: Suppose the first variant to be false, that is $\Omega_{t_0} = \Omega_{t_1}$ for some $0 \le t_0 < t_1$. Call $\varepsilon := (t_1 - t_0)$. We shall show that $\Omega_{t_0} = \Omega_{t_0+n\cdot\varepsilon}$ for all $n \in \mathbb{N}$, which would prove the claim. Indeed, suppose $x \in \Omega_{t_0}$, then $(G_t(x))_{t=0}^{t_1} \subseteq K^o$ and thus $(G_t(G_\varepsilon(x)))_{t=0}^{t_0} \subseteq K^o$ which implies $G_\varepsilon(x) \in \Omega_{t_0}$. This again means that $G_\varepsilon(x) \in \Omega_{t_1}$, that is $(G_t(x))_{t=\varepsilon}^{t_1+\varepsilon} \subseteq K^o$ and therefore $(G_t(x))_{t=0}^{t_0+2\varepsilon} \subseteq K^o$, thus $x \in \Omega_{t_0+2\varepsilon}$. The rest follows by induction.

Claim: $\Omega = \Omega_{t_0}$.

Proof: Suppose the contrary, then $\Omega_{t_0} \supsetneq \Omega_{t_1}$ for all $0 \le t_0 < t_1$. We shall show that $G_t(A) \nsubseteq A$ for some $t \ge 0$, a contradiction!

Choose $x_n \in \Omega_n \setminus \Omega_{n+1}$ and set $y_n := G_n(x_n)$ for every $n \in \mathbb{N}$. Consider $(y_n)_n$ as a net. Then by compactness of K, there exists a subnet (not necessarily subsequence!) $(y_{n(k)})_{k\in I}$ of $(y_n)_n$ that converges towards some $y \in K$. Now each y_n belongs to the intersection $\bigcap_{0\le t\le n} G_n(K)$, since $y = G_n(x_n) = G_t(G_{n-t}(x_n)) \in G_t(K)$ for each $0 \le t \le n$. Otherwise said, for each $t \ge 0$ the sequence $(y_n)_n$ is eventually in $G_t(K)$, a property shared by the subnet $(y_{n(k)})_{k\in I}$. Since G_t is continuous and X Hausdorff, each $G_t(K)$ is compact and therefore closed. Thus, the limit y lies within each of the $G_t(K)$. As $\bigcap_{t\ge 0} G_t(K) \subseteq A$, we find that $y \in A$. On the other hand $x_n \notin \Omega_{n+1}$, implying that $G_{\varepsilon_n}(y_n) \notin K^o$ for some $0 \le \varepsilon_n \le 1$ for every $n \in \mathbb{N}$. As $[0,1]$ is compact, we can suppose $(\varepsilon_{n(k)})_{k\in I}$ to be converging towards some $t \in [0,1]$. Thus the subnet $(t_{n(k)}, y_{n(k)})_{k\in I}$ converges towards (t, y). Since $G : \mathbb{R}_+ \times X \to X$ is continuous, we find that $G_t(y) = \lim_{k\in I} G_{t_{n(k)}}(y_{n(k)}) \notin K^o$, since $(K^o)^c$ is closed. This implies $G_t(y) \notin A$, since $A \subseteq K^o$. Together with $y \in A$, this is a contradiction!

Thus $\Omega = \Omega_{t_0}$ for some $t_0 \ge 0$, that is Ω_{t_0} is indeed an open neighborhood of A.

□

Remark: The proof actually shows that such a forward invariant neighborhood Ω is given by the set of all start-points in K^o whose future orbits stay within K^o. This so constructed Ω is the greatest forward invariant neighborhood of A included within K^o. The proof also reveals that there exists some $t_0 \ge 0$, such that Ω is characterized as being the set of all start-points in K^o whose orbits stay within K^o up to time t_0.

1.3.5 Lemma: Neighborhoods of attraction in trapping neighborhoods

Let (X, G) be a real-time, continuous dynamical system on the Hausdorff space X and $A \subseteq X$ some forward invariant, compact set. Let $K \subseteq X$ be a compact neighborhood of A such that $\bigcap_{t\ge 0} G_t(K) \subseteq A$. Then every forward invariant neighborhood Ω of A included in K is a neighborhood of attraction for A.

Proof: Let B be some neighborhood of A and $x_0 \in \Omega$. We show that the future orbit $(x(t))_{t\ge 0} := (G_t(x_0))_{t\ge 0}$ is eventually in B. We can assume that $B \subseteq K$. Now since B is included in a compact and $A \subseteq B^o$ is compact and X Hausdorff, there exists a compact neighborhood $\widetilde{K}$ of A included in B. It satisfies $\bigcap_{t\ge 0} G_t(\widetilde{K}) \subseteq \bigcap_{t\ge 0} G_t(K) \subseteq A$ and contains therefore by lemma 1.3.4 a forward invariant neighborhood $\widetilde{B}$ of A. To sum it up, we can suppose B to be already forward invariant.
It thus suffices to show that $(x(t))_{t\ge 0}$ passes by B. Suppose the contrary, that is $x(t) \notin B$ for all $t \ge 0$. Since K is compact, the orbit $(x(t))_{t\ge 0}$ has a cluster point x in K.
Since Ω is forward invariant, the sets $G_t(\Omega)$ are decreasing with increasing t. Thus $x(t) \in G_{t_0}(\Omega)$ for all $0 \le t_0 \le t$, that is, for every $t_0 \ge 0$ the orbit $(x(t))_{t\ge 0}$ is eventually in $G_{t_0}(\Omega)$. Since $\overline{\Omega}$ is compact, being closed within the compact set K, each $G_{t_0}(\overline{\Omega})$ is compact and thus closed. Thus the cluster point x is within each $G_{t_0}(\overline{\Omega})$ and consequently in the intersection $\bigcap_{t\ge 0} G_t(\overline{\Omega}) \subseteq \bigcap_{t\ge 0} G_t(K) \subseteq A$. This is a contradiction to the orbit $(x(t))_{t\ge 0}$ not passing by B!

□

1.3.6 Corollary for trapping neighborhoods

Let (X, G) be a real-time, continuous dynamical system on the locally compact Hausdorff space X and $A \subseteq X$ some forward invariant, compact set. Let $U \subseteq X$ be a neighborhood of A such that $A \supseteq \bigcap_{t\ge 0} G_t(U)$. Then there exists a forward invariant, open (compact) neighborhood of attraction Ω of A such that $A \subseteq \Omega \subseteq U^o$.

Proof: Since A is compact and the space locally compact, Hausdorff, there exists a compact neighborhood K of A such that $A \subseteq K \subseteq U^o$. Since $\bigcap_{t\ge 0} G_t(K) \subseteq \bigcap_{t\ge 0} G_t(U) \subseteq A$, by lemma 1.3.4 there exists a forward invariant, open neighborhood Ω of A such that $A \subseteq \Omega \subseteq K^o$. The closure $\overline{\Omega} \subseteq K$ of Ω in X is compact and by lemma 1.1.8(1) forward invariant. By lemma 1.3.5, $\overline{\Omega}$ (and thus Ω) is a neighborhood of attraction for A.

□

1.3.7 Theorem: Attractors as stable sets

Let (X, G) be a real-time, continuous dynamical system on the locally compact Hausdorff space X. Then every attractor $A \subseteq X$ is an invariant, Lyapunov table, locally attracting set.

Proof: By definition 1.3.1, every attractor is compact and invariant. Let U be a forward invariant neighborhood of A such that $A = \bigcap_{t \geq 0} G_t(U)$. By corollary 1.3.6, A has a neighborhood of attraction, thus is locally attracting. Left to show is that A is Lyapunov stable. Let B be some neighborhood of A and suppose w.l.o.g. that $B \subseteq U$. Then $\bigcap_{t \geq 0} G_t(B) \subseteq A$ and by corollary 1.3.6 there exists a forward invariant neighborhood Ω of A such that $A \subseteq \Omega \subseteq B$.

□

1.3.8 Lemma on decreasing compact sets

Let X, Y be two topological spaces and X Hausdorff. Let $(B_t)_{t \geq 0}$ be a family of compact subsets of X such that $B_{t_0} \supseteq B_{t_1}$ for every $0 \leq t_0 \leq t_1$. Let $f : X \to Y$ be continuous. Then $f\left(\bigcap_{t \geq 0} B_t\right) = \bigcap_{t \geq 0} f(B_t)$.

Proof: The inclusion $f\left(\bigcap_{t \geq 0} B_t\right) \subseteq \bigcap_{t \geq 0} f(B_t)$ is trivial. Now let $y \in \bigcap_{t \geq 0} f(B_t)$, then $y = f(x_t)$ for some $x_t \in B_t$ for all $t \geq 0$. As $(B_t)_t$ is decreasing, $x_t \in B_\tau$ for every $0 \leq \tau \leq t$. Since B_0 is compact, there exists a subnet $(x_t)_{t \in T}$ of $(x_t)_{t \geq 0}$ that converges towards some $x \in X$. Since $(x_t)_{t \in T}$ is for every $t_0 \geq 0$ eventually within the (closed) B_{t_0}, its limit x is also within B_{t_0}. Thus $x \in \bigcap_{t \geq 0} B_t$. By continuity of f, one has $f(x) = \lim_{T \ni t \to \infty} f(x_t) = \lim_{T \ni t \to \infty} y = y$.

□

1.3.9 Theorem: Characterization of attractors

Let (X, G) be a real-time, continuous, space-open dynamical system on the locally compact Hausdorff space X. For any compact set $\emptyset \neq A \subseteq X$ the following are equivalent:

1. A is an attractor.
2. A is an invariant, Lyapunov stable and locally attracting set.
3. There exists a compact, forward invariant neighborhood K of A such that $\bigcap_{t \geq 0} G_t(K) = A$.
4. A is invariant and there exists a neighborhood U of A such that $\bigcap_{t \geq 0} G_t(U) \subseteq A$. In that case actually equality "=" holds.

Proof:

(1)⇒(4): Follows from the definition of an attractor.

(1)⇒(2): See theorem 1.3.7.

(2)⇒(1): See theorem 1.3.2.

(4)⇒(3): The equality holds since $G_t(A) \supseteq A$ and thus $G_t(U) \supseteq A$ for all $t \geq 0$. By corollary 1.3.6 there exists a compact, forward invariant neighborhood K of A such that $A \subseteq K \subseteq U$ and thus $\bigcap_{t \geq 0} G_t(K) \subseteq A$. Since $G_t(A) \supseteq A$ and thus $G_t(K) \supseteq A$ for all $t \geq 0$, one has actually $\bigcap_{t \geq 0} G_t(K) = A$.

(3)⇒(1): For every $t_0 \geq 0$ one has

$$G_{t_0}(A) \subseteq \bigcap_{t \geq 0} \underbrace{G_{t_0}(G_t(K))}_{\substack{\subseteq G_t(K) \\ \text{by forw. invariance}}} \subseteq \bigcap_{t \geq 0} G_t(K) = A, \tag{1.7}$$

hence A is forward invariant. On the other hand, each $G_t(K)$ is compact and the sequence $(G_t(K))_{t \geq 0}$ decreasing in t, so that by lemma 1.3.8

$$G_{t_0}(A) \supseteq \bigcap_{t \geq 0} G_{t_0}(G_t(K)) = \bigcap_{t \geq t_0} G_t(K) \supseteq \bigcap_{t \geq 0} G_t(K) = A. \tag{1.8}$$

Hence, A is invariant. That A is an attractor now follows from the definition.

□

1.3.10 Theorem: Existence of attractors

Let (X, G) be a real-time, space-continuous dynamical system on the Hausdorff space X. Suppose $K \subseteq X$ is a compact, forward invariant set such that $G_t(K) \subseteq K^o$ for some $t \geq 0$. Then the limit $A := \bigcap_{t\geq 0} G_t(K)$ is an attractor.

Proof: Since every $G_t(K)$ is compact and X is Hausdorff, their intersection A is also compact. Since $(G_t(K))_{t\geq 0}$ is a system of non-empty compacts, decreasing with increasing t, it satisfies the finite intersection property and has thus non-empty intersection $A \neq \emptyset$. By assumption $A \subseteq K^o$, hence K is a neighborhood of A. For every $t_0 \geq 0$ one has

$$G_{t_0}(A) \subseteq \bigcap_{t\geq 0} \underbrace{G_{t_0}(G_t(K))}_{\substack{\subseteq G_t(K) \\ \text{by forw. invariance}}} \subseteq \bigcap_{t\geq 0} G_t(K) = A, \tag{1.9}$$

hence A is forward invariant. On the other hand, each $G_t(K)$ is compact and the sequence $(G_t(K))_{t\geq 0}$ decreasing in t, so that by lemma 1.3.8

$$G_{t_0}(A) \supseteq \bigcap_{t\geq 0} G_{t_0}(G_t(K)) = \bigcap_{t\geq t_0} G_t(K) \supseteq \bigcap_{t\geq 0} G_t(K) = A. \tag{1.10}$$

Hence, A is invariant. Therefore, A satisfies all axioms in 1.3.1 and is an attractor.

□

1.3.11 Corollary: Existence of attractors

Every real-time, space-continuous dynamical system on a compact Hausdorff space X has a global attractor.

Proof: The space itself is compact, forward invariant and satisfies $G_t(X) \subseteq X = X^o$. Therefore $A := \bigcap_{t\geq 0} G_t(X)$ is an attractor by theorem 1.3.10.

□

1.3.12 Example: Non-minimal attractors

We shall present an example of a real-time, continuous, space-open invertible dynamical system (X, G) on a compact metric space that has a countably infinite number of attractors, none of which is minimal. We consider the unit-disc $X := \{z \in \mathbb{C} : |z| \leq 1\}$ and the concentric discs $A_n := \left\{z \in \mathbb{C} : |z| \leq \frac{1}{n}\right\}$ for $n \in \mathbb{N}$. We consider the flow corresponding to the differential equation

$$\frac{dz}{dt} = \left\{ \frac{z}{|z|} \cdot \left(|z| - \frac{1}{n}\right) \cdot \left(|z| - \frac{1}{n+1}\right) \cdot n \cdot (n+1) \quad : \frac{1}{n+1} \leq |z| \leq \frac{1}{n}, n \in \mathbb{N} \right. \tag{1.11}$$

having the solution

$$G_t(z_0) := \begin{cases} \frac{z}{z_0} \cdot \frac{\alpha_n(|z_0|) - e^t}{n \cdot \alpha_n(|z_0|) - (n+1) \cdot e^t} & : \frac{1}{n+1} \leq |z_0| \leq \frac{1}{n},\ n \in \mathbb{N} \\ 0 & : |z_0| = 0 \end{cases} \tag{1.12}$$

for $t \in \mathbb{R}$, $z_0 \in X$, whereas

$$\alpha_n(r) := \frac{(n+1) \cdot r - 1}{n \cdot r - 1} \tag{1.13}$$

for $\frac{1}{n+1} \leq r \leq \frac{1}{n}$ and $n \in \mathbb{N}$. Note how the flow is continuous and invertible, thus space-open and space-closed. It corresponds to a radially uniform flow inwards, halting exactly on each of the circles $\partial A_n = \left\{z \in X : |z| = \frac{1}{n}\right\}$ and the origin $A_\infty := \{0\}$. Each of the discs A_n (and their interior A_n^o, $n \in \mathbb{N}$) is invariant to the flow and all points strictly between two circles ∂A_n, ∂A_{n+1} converge towards (but never arrive at) the inner one A_{n+1}

under the forward action of G. Thus, each A_n $(2 \leq n \in \mathbb{N})$ is an attractor with $A_n = \bigcap_{t\geq 0} G_t(A_{n-\frac{1}{2}})$, whereas $A_{n-\frac{1}{2}} := \left\{z \in \mathbb{C} : |z| \leq \frac{1}{n-\frac{1}{2}}\right\}$ is a forward invariant (but not invariant!) neighborhood of A_n. The space $X = A_1$ itself is a global attractor, that is $X = \bigcap_{t\geq 0} G_t(X)$. Notice how none of these attractors is minimal.

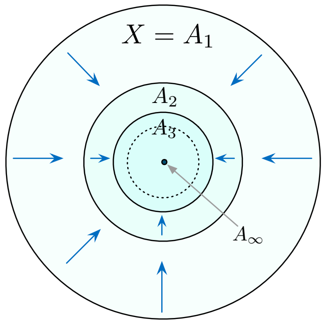

Figure 1.1: On the construction of a dynamical system with infinitely many nested attractors, none of which is minimal. Note that each disc around the origin is forward invariant to the flow, but only the discs A_n $(n \in \mathbb{N} \cup \{\infty\})$, as well as their borders ∂A_n and interiors A_n^o, are invariant.

The origin A_∞ is a fixed point. It is Lyapunov stable, since each of its neighborhoods contains a sufficiently small invariant disc A_n. It is nonetheless not locally attracting, since every point $z \neq 0$ is *for ever captured* between two consecutive circles ∂A_n, ∂A_{n+1}.

1.4 Attractors in topologically transitive systems

1.4.1 Lemma: Characterization of topologically transitive systems

Let (X, G) be a real-time dynamical system on the topological space X. Then of the following, (1) and (2) are equivalent and implied by (3). If the system is space-open, all three statements are equivalent.

1. The system is topologically transitive.
2. Every forward invariant neighborhood is dense in X.
3. The interior of every forward invariant neighborhood is dense in X.

Proof:

(1)⇒(2): If $U \subseteq X$ is a forward invariant neighborhood, then every other non-empty open set V is intersected by $G_t(U^o)$ for some $t \geq 0$. But $G_t(U^o) \subseteq U$, which shows that U is dense in X.

(2)⇒(1): Let $U, V \subseteq X$ be two non-empty open sets. Then $\bigcup_{t\geq 0} G_t(U)$ is forward invariant and a neighborhood, thus dense in X. It therefore intersects V, which implies $G_t(U) \cap V \neq \emptyset$ for some $t \geq 0$.

(3)⇒(2): Trivial.

(1)⇒(3): Suppose (X, G) to be space-open and topologically transitive. If $U \subseteq X$ is a forward invariant neighborhood, then every other non-empty open set V is intersected by $G_t(U^o)$ for some $t \geq 0$. But $G_t(U^o) \subseteq U^o$ since G_t is an open mapping, which shows that U^o is dense in X.

□

1.4.2 Theorem: Attractors in topologically transitive systems

Let (X, G) be a real-time, space-open, topologically transitive dynamical system on a locally compact Hausdorff space X. Then the only possible attractor is the space itself, in which case X has to be compact.

Proof: We start by showing the density of any existing attractor A in X. Let U be a forward invariant neighborhood of A such that $A = \bigcap_{t\geq 0} G_t(U)$. By lemma 1.1.8(1), the interior U^o is forward invariant. It satisfies $\bigcap_{t\geq 0} G_t(U^o) \subseteq A$. For every $t \geq 0$, the set $G_t(U^o)$ is open since G_t is an open mapping and forward invariant since U^o is forward invariant. Lemma 1.4.1 therefore implies that $G_t(U^o)$ be dense in A. Thus, $\bigcap_{t\geq 0} G_t(U^o) = \bigcap_{n\in\mathbb{N}} G_n(U^o)$ is a countable intersection of dense, open sets. As the space is locally compact Hausdorff, by Baire this intersection, and thus A, is dense in X.
The density and compactness of A within the Hausdorff space X implies that it is in fact equal to the whole space, the latter thus being compact.

□

1.4.3 Corollary about topologically transitive attractors

Let (X, G) be a real-time, space-open dynamical system on a Hausdorff space X. Then any topologically transitive attractor is minimal.

Proof: Suppose $A \subseteq X$ to be a topologically transitive attractor containing another attractor $\widetilde{A} \subseteq A$. By remark 1.3.1(v) $\widetilde{A}$ is also an attractor for the restricted system $(A, G\big|_A)$. The latter satisfies the conditions in theorem 1.4.2, by which $\widetilde{A}$ has to be the whole space A.

□

A Appendix

A.0.4 Lemma: Separating compacts in Hausdorff spaces

Let X be a Hausdorff space and $K_1, K_2 \subseteq X$ disjoint compact sets. Then there exist disjoint open sets $U_1, U_2 \subseteq X$ such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$.

Proof: See [4].

A.0.5 Lemma: Necessary condition for locally compact spaces

Let X be a topological space and $A_n \subseteq X$ closed subsets, such that $X = \bigcup_{n=1}^{\infty} A_n$. Let $\emptyset \neq A \subseteq X$ be such that, for every $n \in \mathbb{N}$ one has $A \subseteq \overline{A \setminus A_n}$. Then X can not be locally compact Hausdorff.

Proof: See [6], Appendix 2.45.

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